

# Dirac structures of omni-Lie algebroids \*

Zhuo Chen<sup>1</sup>, Zhangju Liu<sup>2</sup> and Yunhe Sheng<sup>3</sup>

<sup>1</sup>Department of Mathematics,

Tsinghua University, Beijing 100084, China

<sup>2</sup>Department of Mathematics and LMAM

Peking University, Beijing 100871, China

<sup>3</sup>Department of Mathematics

Jilin University, Changchun 130012, Jilin, China

email: <sup>1</sup>chenzhuott@gmail.com, <sup>2</sup>liuzj@pku.edu.cn, <sup>3</sup>ysheng888@gmail.com

## Abstract

The generalized Courant algebroid structure attached to the direct sum  $\mathcal{E} = \mathfrak{D}E \oplus \mathfrak{J}E$  for a vector bundle  $E$  is called an omni-Lie algebroid, as it is reduced to the omni-Lie algebra introduced by A. Weinstein if the base manifold is a point. A Dirac structure in  $\mathcal{E}$  is necessarily a Lie algebroid associated with a representation on  $E$ . We study the geometry underlying these Dirac structures in the light of reduction theory. In particular, we prove that there is a one-to-one correspondence between reducible Dirac structures of  $\mathcal{E}$  and projective Lie algebroids in  $\mathcal{T} = TM \oplus E$ ; we establish the relation between the normalizer  $N_L$  of a reducible Dirac structure  $L$  and the derivation algebra  $\text{Der}(\mathfrak{b}(L))$  of the projective Lie algebroid  $\mathfrak{b}(L)$ ; we study the cohomology group  $H^\bullet(L, \rho_L)$  and the relation between  $N_L$  and  $H^1(L, \rho_L)$ ; we describe Lie bialgebroids using the adjoint representation and the deformation of a Dirac structure, which is related with  $H^2(L, \rho_L)$ .

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# 1 Introduction

Lie algebroids (and local Lie algebras in the sense of Kirillov [14]) are generalizations of Lie algebras that naturally appear in Poisson geometry (and its variations, e.g., Jacobi manifolds in the sense of Lichnerowicz [17])(see [21] for a detailed description of this subject). Courant algebroids are combinations of Lie algebroids and quadratic Lie algebras. It was originally introduced in [8] by T. Courant where he first called them Dirac manifolds, and then were re-named after him in [20] (see also an alternate definition [27]) by Liu, Weinstein and Xu to describe the double of a Lie bialgebroid. Recently, several applications of Courant algebroids and Dirac structures have been found in different fields, e.g., Manin pairs and moment maps [1], [4]; generalized complex structures [3], [10];  $L_\infty$ -algebras and symplectic supermanifolds [24]; gerbes [26] as well as BV algebras and topological field theories [12], [25].

Motivated by an integrability problem of the Courant bracket, A. Weinstein gives a linearization of the Courant bracket at a point [31], which is studied from several aspects recently ([3, 13, 23, 28]). Since Dirac structures of Courant algebroids are natural providers of Lie algebroids and A. Weinstein has shown that an omni-Lie algebra structure can encode all Lie algebra structures, the next step is, logically, to find out candidates that could encode all Lie algebroid structures. In a recent work [6], we have given a definitive answer to this question.

Let us first review the contents of [6]. A generalized Courant algebroid structure is defined on the direct sum bundle  $\mathfrak{D}E \oplus \mathfrak{J}E$ , where  $\mathfrak{D}E$  and  $\mathfrak{J}E$  are the gauge Lie algebroid and the jet bundle of a vector bundle  $E$  respectively. Such a structure is called an *omni-Lie algebroid* since it reduces to the omni-Lie algebra introduced by A. Weinstein if the base manifold is a point [31].

It is well known that the theory of Dirac structures has wide and deep applications in both mathematics and physics (e.g., [2], [5], [9], [10], [11], [30]). In [6], only some special Dirac structures were studied and it is proved that there is a one-to-one correspondence between Dirac structures coming from bundle maps  $\mathfrak{J}E \rightarrow \mathfrak{D}E$  and Lie algebroid (local Lie algebra) structures on  $E$  when  $\text{rank}(E) \geq 2$  ( $E$  is a line bundle). In other words, Dirac structures that are graphs of maps actually underlines the geometric objects of Lie algebroids, or local Lie algebras.

As a continuation of [6], the present paper explores what a general Dirac structure of the omni-Lie algebroid would encode. As we shall see, for a vector space  $V$ , Dirac structures in the omni-Lie algebra  $\mathfrak{gl}(V) \oplus V$  come from Lie algebra structures on subspaces of  $V$  (this coincides with Weinstein's result [31]). For a vector bundle  $E$  over  $M$ , Dirac structures in the omni-Lie algebroid  $\mathcal{E} = \mathfrak{D}E \oplus \mathfrak{J}E$  turn out to be more complicated than that of omni-Lie algebras. The key concept we need is that of a projective Lie algebroid — a subbundle  $A \subset \mathcal{T} = TM \oplus E$ , which is equipped with a Lie algebroid structure such that the anchor is the projection from  $A$  to  $TM$ . A Dirac structure  $L \subset \mathcal{E}$  is called reducible if  $\mathbf{b}(L)$  is a regular subbundle of  $\mathcal{T}$ . We shall see that any Dirac structure is reducible if  $\text{rank}(E) \geq 2$  (Lemma 3.1).

The main result is Theorem 3.7, which claims a one-to-one correspondence between reducible Dirac structures in  $\mathcal{E}$  and projective Lie algebroids in  $\mathcal{T}$ . In fact, the projection of a reducible Dirac structure  $L$  to  $\mathcal{T}$  yields a projective Lie algebroid  $\mathbf{b}(L)$  and, conversely, a projective Lie algebroid  $A \subset \mathcal{T}$  can be uniquely lifted to a Dirac structure  $L^A$  by means of a connection in  $E$ .

Furthermore, using the falling operator  $(\cdot)_\bullet$ , we establish a connection between the deriva-

tion algebra  $\text{Der}(A)$  of a projective Lie algebroid  $A$  and the normalizer  $N_{L^A}$  of the corresponding lifted Dirac structure  $L^A$ . We prove that, for any  $X \in N_{L^A}$ ,  $X_\bullet \in \text{Der}(A)$ . Conversely, any  $\delta \in \text{Der}(A)$  can be lifted to an element in  $N_{L^A}$ . Another observation is that, to any Dirac structure  $L \subset \mathcal{E}$ , there associates a representation of  $L$  on  $E$ , namely  $\rho_L : L \rightarrow \mathfrak{D}E$  (Proposition 2.5). So there is an associated cohomology group  $H^\bullet(L, \rho_L)$ . We will see that the normalizer of  $L$  is related with  $H^1(L, \rho_L)$  and the deformation of  $L$  is related with  $H^2(L, \rho_L)$ .

This paper is organized as follows. In Section 2 we recall the basic properties of omni-Lie algebroids. In Section 3, we state the main result of this paper — the correspondence between reducible Dirac structures and projective Lie algebroids. In Section 4, several interesting examples are discussed. In Section 5, we study the relation between the normalizer of a reducible Dirac structure and Lie derivations. In Section 6, we give some applications of the related cohomologies of Dirac structures.

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## 2 Omni-Lie Algebroids

We use the following convention throughout the paper:  $E \rightarrow M$  denotes a vector bundle  $E$  over a smooth manifold  $M$  (we assume that  $E$  is not a zero bundle),  $d : \Omega^\bullet(M) \rightarrow \Omega^{\bullet+1}(M)$  the usual deRham differential of forms and  $m$  an arbitrary point in  $M$ . By  $\mathcal{T}$  we denote the direct sum  $TM \oplus E$  and use  $pr_{TM}$ ,  $pr_E$ , respectively, to denote the projection from  $\mathcal{T}$  to  $TM$  and  $E$ .

First, we briefly review the notion of omni-Lie algebroids defined in [6], which generalizes omni-Lie algebras defined by A. Weinstein in [31]. Given a vector bundle  $E$ , let  $\mathfrak{J}E$  be the (1-)jet bundle of  $E$  ([22]), and  $\mathfrak{D}E$  the gauge Lie algebroid of  $E$  ([21]). These two vector bundles associate, respectively, with the jet sequence:

$$0 \rightarrow \text{Hom}(TM, E) \xrightarrow{\mathfrak{e}} \mathfrak{J}E \xrightarrow{\mathfrak{p}} E \rightarrow 0, \quad (1)$$

and the Atiyah sequence:

$$0 \rightarrow \mathfrak{gl}(E) \xrightarrow{\mathfrak{i}} \mathfrak{D}E \xrightarrow{\alpha} TM \rightarrow 0. \quad (2)$$

The embedding maps  $\mathfrak{e}$  and  $\mathfrak{i}$  in the above two exact sequences will be ignored when there is no risk of confusion. It is well known that  $\mathfrak{D}E$  is a transitive Lie algebroid over  $M$ , with the anchor  $\alpha$  as above ([15]). The  $E$ -duality between two vector bundles is defined as follows.

**Definition 2.1.** *Let  $A$ ,  $B$  and  $E$  be vector bundles over  $M$ . We say that  $B$  is an  $E$ -dual bundle of  $A$  if there is a  $C^\infty(M)$ -bilinear  $E$ -valued pairing  $\langle \cdot, \cdot \rangle_E : A \times_M B \rightarrow E$  which is nondegenerate, that is, the map  $a \mapsto \langle a, \cdot \rangle_E$  is an embedding of  $A$  into  $\text{Hom}(B, E)$ , and similarly for the  $B$ -entry.*

An important result in [6] is that  $\mathfrak{J}E$  is an  $E$ -dual bundle of  $\mathfrak{D}E$  with some nice properties. In fact, we have a nondegenerate  $E$ -pairing  $\langle \cdot, \cdot \rangle_E$  between  $\mathfrak{J}E$  and  $\mathfrak{D}E$ :

$$\langle \mu, \mathfrak{d} \rangle_E = \langle \mathfrak{d}, \mu \rangle_E \triangleq \mathfrak{d}u, \quad \forall \mu = [u]_m \in \mathfrak{J}E, \quad u \in \Gamma(E), \quad \mathfrak{d} \in \mathfrak{D}E.$$

Moreover, this pairing is  $C^\infty(M)$ -linear and satisfies the following properties:

$$\begin{aligned}\langle \mu, \Phi \rangle_E &= \Phi \circ \mathbb{P}(\mu), \quad \forall \Phi \in \mathfrak{gl}(E), \mu \in \mathfrak{J}E; \\ \langle \eta, \mathfrak{d} \rangle_E &= \eta \circ \alpha(\mathfrak{d}), \quad \forall \eta \in \text{Hom}(TM, E), \mathfrak{d} \in \mathfrak{D}E.\end{aligned}$$

An equivalent expression is that we can define  $\mathfrak{J}E$  by  $\mathfrak{D}E$ ,

$$\mathfrak{J}E \cong \{ \nu \in \text{Hom}(\mathfrak{D}E, E) \mid \nu(\Phi) = \Phi \circ \nu(\mathbf{1}_E), \quad \forall \Phi \in \mathfrak{gl}(E) \} \subset \text{Hom}(\mathfrak{D}E, E).$$

Conversely,  $\mathfrak{D}E$  is also determined by  $\mathfrak{J}E$ :

$$\mathfrak{D}E \cong \{ \delta \in \text{Hom}(\mathfrak{J}E, E) \mid \exists x \in TM, \text{ s.t. } \delta(\eta) = \eta(x), \quad \forall \eta \in \text{Hom}(TM, E) \}.$$

For a Lie algebroid  $(\mathcal{A}, [\cdot, \cdot], \alpha)$  over  $M$ , a *representation* of  $\mathcal{A}$  on a vector bundle  $E \rightarrow M$  is a Lie algebroid morphism  $\mathcal{L} : \mathcal{A} \rightarrow \mathfrak{D}E$ . We may also refer to  $E$  as an  $\mathcal{A}$ -module. To such a representation, there associates a cochain complex  $\sum_{i \geq 0} \Omega^i(\mathcal{A}, E) = \sum_{i \geq 0} \Gamma(\text{Hom}(\wedge^i \mathcal{A}, E))$  with the coboundary operator:

$$d_{\mathcal{A}} : \Omega^\bullet(\mathcal{A}, E) \rightarrow \Omega^{\bullet+1}(\mathcal{A}, E),$$

defined in a similar fashion as that of the deRham differential [21]. Since  $\mathfrak{D}E$  is a Lie algebroid and  $E$  is a natural  $\mathfrak{D}E$ -module, we have the cochain complex:

$$\Omega^\bullet(\mathfrak{D}E, E) = \Gamma(\text{Hom}(\wedge^\bullet \mathfrak{D}E, E))$$

with the coboundary operator:

$$\mathfrak{d} : \Omega^\bullet(\mathfrak{D}E, E) \rightarrow \Omega^{\bullet+1}(\mathfrak{D}E, E). \quad (3)$$

Note that,  $\forall u \in \Gamma(E)$ ,  $\mathfrak{d}u \in \Omega^1(\mathfrak{D}E, E)$  is a section of  $\mathfrak{J}E$  and we have a formula:

$$\mathfrak{d}(fu) = f\mathfrak{d}u + \mathfrak{d}f \otimes u, \quad \forall f \in C^\infty(M), u \in \Gamma(E).$$

The section space  $\Gamma(\mathfrak{J}E)$  is an invariant subspace of the Lie derivative  $\mathfrak{L}_{\mathfrak{d}}$  for any  $\mathfrak{d} \in \Gamma(\mathfrak{D}E)$ . Here  $\mathfrak{L}_{\mathfrak{d}}$  is defined by the Leibniz rule as follows:

$$\langle \mathfrak{L}_{\mathfrak{d}}\mu, \mathfrak{d}' \rangle_E \triangleq \mathfrak{d} \langle \mu, \mathfrak{d}' \rangle_E - \langle \mu, [\mathfrak{d}, \mathfrak{d}']_{\mathfrak{D}} \rangle_E, \quad \forall \mu \in \Gamma(\mathfrak{J}E), \mathfrak{d}' \in \Gamma(\mathfrak{D}E).$$

**Definition 2.2.** [6] We call the quadruple  $(\mathcal{E}, \{\cdot, \cdot\}, (\cdot, \cdot)_E, \rho)$  an *omni-Lie algebroid*, where  $\mathcal{E} = \mathfrak{D}E \oplus \mathfrak{J}E$ ,  $\rho$  is the projection from  $\mathcal{E}$  to  $\mathfrak{D}E$ , the bracket  $\{\cdot, \cdot\} : \Gamma(\mathcal{E}) \times \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E})$  is defined by

$$\{\mathfrak{d} + \mu, \mathfrak{r} + \nu\} \triangleq [\mathfrak{d}, \mathfrak{r}]_{\mathfrak{D}} + \mathfrak{L}_{\mathfrak{d}}\nu - \mathfrak{L}_{\mathfrak{r}}\mu + \mathfrak{d} \langle \mu, \mathfrak{r} \rangle_E,$$

and  $(\cdot, \cdot)_E$  is a nondegenerate symmetric  $E$ -valued 2-form on  $\mathcal{E}$  defined by:

$$(\mathfrak{d} + \mu, \mathfrak{r} + \nu)_E \triangleq \frac{1}{2}(\langle \mathfrak{d}, \nu \rangle_E + \langle \mathfrak{r}, \mu \rangle_E),$$

for any  $\mathfrak{d}, \mathfrak{r} \in \mathfrak{D}E$ ,  $\mu, \nu \in \mathfrak{J}E$ .

**Theorem 2.3.** [6] An omni-Lie algebroid satisfies the following properties,  $\forall X, Y, Z \in \Gamma(\mathcal{E})$ ,  $f \in C^\infty(M)$ :

- 1)  $(\Gamma(\mathcal{E}), \{\cdot, \cdot\})$  is a Leibniz algebra,
- 2)  $\rho\{X, Y\} = [\rho(X), \rho(Y)]_{\mathfrak{D}}$ ,
- 3)  $\{X, fY\} = f\{X, Y\} + (\alpha \circ \rho(X))(f)Y$ ,
- 4)  $\{X, X\} = \mathfrak{d}(X, X)_E$ ,
- 5)  $\rho(X)(Y, Z)_E = (\{X, Y\}, Z)_E + (Y, \{X, Z\})_E$ .

From these, it is easy to obtain the following equalities:

$$\{fX, Y\} = f\{X, Y\} - (\alpha \circ \rho(Y))(f)Y + 2df \otimes (X, Y)_E, \quad (4)$$

$$\{X, Y\} + \{Y, X\} = 2\mathfrak{d}(X, Y)_E. \quad (5)$$

For a subbundle  $S \subset \mathcal{E}$ , we denote

$$S^\perp = \{X \in \mathcal{E} \mid (X, s)_E = 0, \quad \forall s \in S\}.$$

We call  $S$  isotropic with respect to  $(\cdot, \cdot)_E$  if  $S \subset S^\perp$ .

**Definition 2.4.** [6] *A Dirac structure in the omni-Lie algebroid  $\mathcal{E}$  is a maximal isotropic<sup>1</sup> subbundle  $L \subset \mathcal{E}$  such that  $\{\Gamma(L), \Gamma(L)\} \subset \Gamma(L)$ .*

**Proposition 2.5.** [6] *A Dirac structure  $L$  is necessarily a Lie algebroid with the restricted bracket and the anchor  $\alpha \circ \rho$ . Moreover,  $\rho_L = \rho|_L : L \rightarrow \mathfrak{D}E$  is a representation of  $L$  on  $E$ .*

For  $\mathcal{T} = TM \oplus E$ , we have the standard decomposition

$$\text{Hom}(\mathcal{T}, E) = \mathfrak{gl}(E) \oplus \text{Hom}(TM, E).$$

The following exact sequence will be referred as the omni-sequence of  $E$ .

$$0 \longrightarrow \text{Hom}(\mathcal{T}, E) \xrightarrow{\mathbf{a}} \mathcal{E} \xrightarrow{\mathbf{b}} \mathcal{T} \longrightarrow 0, \quad (6)$$

where the maps  $\mathbf{a}$  and  $\mathbf{b}$  are defined, respectively, by

$$\mathbf{a}(\Phi + \eta) = \mathfrak{i}(\Phi) + \mathfrak{e}(\eta), \quad \forall \Phi \in \mathfrak{gl}(E), \quad \eta \in \text{Hom}(TM, E);$$

$$\mathbf{b}(\mathfrak{d} + \mu) = \alpha(\mathfrak{d}) + \mathfrak{p}(\mu), \quad \forall \mathfrak{d} \in \mathfrak{D}E, \quad \mu \in \mathfrak{J}E.$$

We regard  $\text{Hom}(\mathcal{T}, E)$  as a subbundle of  $\mathcal{E}$  and omit the embedding  $\mathbf{a}$ . Evidently,  $\text{Hom}(\mathcal{T}, E)$  is a maximal isotropic subbundle of  $\mathcal{E}$ . In fact, it is a Dirac structure of  $\mathcal{E}$  and the bracket is given by

$$\{\alpha, \beta\} = \alpha \circ \beta - \beta \circ \alpha, \quad \forall \alpha, \beta \in \Gamma(\text{Hom}(\mathcal{T}, E)).$$

In particular, if  $\alpha = \Phi + \phi$ ,  $\beta = \Psi + \psi$ , where  $\Phi, \Psi \in \Gamma(\mathfrak{gl}(E))$ ,  $\phi, \psi \in \Gamma(\text{Hom}(TM, E))$ , then

$$\{\Phi, \Psi\} = \Phi \circ \Psi - \Psi \circ \Phi, \quad \{\phi, \psi\} = 0, \quad \{\Phi, \phi\} = \Phi \circ \phi.$$

**Lemma 2.6.** (1) *The subspace  $\Gamma(\text{Hom}(\mathcal{T}, E))$  is a right ideal of  $\Gamma(\mathcal{E})$ .*

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<sup>1</sup>One may prove that  $L$  is maximal isotropic if and only if  $L = L^\perp$ .

(2) For any  $h \in \Gamma(\text{Hom}(\mathcal{T}, E))$ ,  $X \in \Gamma(\mathcal{E})$ , we have

$$\mathbf{b}\{h, X\} = h(\mathbf{b}(X)). \quad (7)$$

Note that (2) implies that the bracket of  $\Gamma(\text{Hom}(\mathcal{T}, E))$  and  $\Gamma(\mathcal{E})$  is fiber-wisely defined.  
**Proof.** For any  $X = \mathfrak{d} + \mu \in \Gamma(\mathcal{E})$  and  $h = \Phi + \mathfrak{h} \in \Gamma(\text{Hom}(\mathcal{T}, E))$ , we have

$$\{\mathfrak{d} + \mu, \Phi + \mathfrak{h}\} = [\mathfrak{d}, \Phi]_{\mathfrak{D}} + \mathfrak{L}_{\mathfrak{d}}\Phi - \mathfrak{L}_{\Phi}\mu + \mathfrak{d}\langle\mu, \Phi\rangle_E.$$

Since

$$\mathbb{P}(-\mathfrak{L}_{\Phi}\mu + \mathfrak{d}\langle\mu, \Phi\rangle_E) = -\Phi\mathbb{P}(\mu) + \langle\mu, \Phi\rangle_E = 0$$

and  $\alpha[\mathfrak{d}, \Phi]_{\mathfrak{D}} = 0$ , we have

$$\{\mathfrak{d} + \mu, \Phi + \mathfrak{h}\} \in \Gamma(\text{Hom}(\mathcal{T}, E)),$$

which implies that  $\Gamma(\text{Hom}(\mathcal{T}, E))$  is a right ideal of  $\Gamma(\mathcal{E})$ .

On the other hand, we have

$$\begin{aligned} \mathbf{b}\{h, X\} &= \mathbf{b}([\Phi, \mathfrak{d}]_{\mathfrak{D}} + \mathfrak{L}_{\Phi}\mu - \mathfrak{L}_{\mathfrak{d}}\mathfrak{h} + \mathfrak{d}\langle\mathfrak{d}, \mathfrak{h}\rangle_E) \\ &= \Phi(\mathbb{P}\mu) + \mathfrak{h}(\alpha\mathfrak{d}) = h(\mathbf{b}(X)), \end{aligned}$$

which completes the proof. ■

### 3 Dirac Structures and Their Reductions

Let us first study some basic properties of maximal isotropic subbundles of  $\mathcal{E}$ . For any subbundle  $Q \subset \mathcal{T}$ , define:

$$Q^0 \triangleq \{h \in \text{Hom}(\mathcal{T}, E) | h(Q) = 0\}.$$

**Lemma 3.1.** *If  $\text{rank}(E) = r$ ,  $\dim(M) = d$ , then for any maximal isotropic subbundle  $L \subset \mathcal{E}$ , we have*

$$\text{rank}(L_m) = (1 - r)\text{rank}(\mathbf{b}(L_m)) + r(d + r), \quad \forall m \in M. \quad (8)$$

*Consequently, if  $r \geq 2$ , both  $\mathbf{b}(L)$  and  $\mathbf{b}(L)^0$  are regular subbundles of, respectively,  $\mathcal{T}$  and  $\mathcal{E}$ . If  $r = 1$ , that is,  $E$  is a line bundle, then  $\text{rank}(L) = d + 1$ .*

**Proof.** Since  $L$  is maximal isotropic, or equivalently,  $L = L^\perp$ , it is not hard to establish the following exact sequence:

$$0 \longrightarrow (\mathbf{b}(L_m))^0 \xrightarrow{\mathbf{a}} L_m \xrightarrow{\mathbf{b}} \mathbf{b}(L_m) \longrightarrow 0. \quad (9)$$

Therefore, we have

$$\begin{aligned} \text{rank}(L_m) &= \text{rank}(\mathbf{b}(L_m)) + \text{rank}(\mathbf{b}(L_m))^0 \\ &= \text{rank}(\mathbf{b}(L_m)) + (r + d - \text{rank}(\mathbf{b}(L_m))) \times r \\ &= (1 - r)\text{rank}(\mathbf{b}(L_m)) + r(d + r). \quad \blacksquare \end{aligned}$$

**Definition 3.2.** For a vector subbundle  $A \subset \mathcal{T}$ , a section  $s : A \longrightarrow \mathcal{E}$  (i.e.  $\mathbf{b} \circ s = \mathbf{1}_A$ ) is called *isotropic* if its image  $s(A) \subset \mathcal{E}$  is isotropic. Two isotropic sections  $s_1$  and  $s_2$  are said to be *equivalent* if  $(s_1 - s_2)(A) \subset A^0$ . The equivalence class of an isotropic section  $s$  is denoted by  $\tilde{s}$ .

**Proposition 3.3.** If  $\text{rank} E \geq 2$ , there is a one-to-one correspondence between maximal isotropic subbundles  $L \subset \mathcal{E}$  and pairs  $(A, \tilde{s})$ , where  $A$  is a subbundle of  $\mathcal{T}$  and  $s : A \rightarrow \mathcal{E}$  is an isotropic section.

For this reason, we call  $(A, \tilde{s})$  the *characteristic pair* of  $L$ , and write  $L = L_{s,A}$ .

**Proof.** Let  $L \subset \mathcal{E}$  be a maximal isotropic subbundle and  $A = \mathbf{b}(L)$ . By Lemma 3.1,  $A$  is a regular subbundle. Any split  $s : A \rightarrow L$  of the corresponding exact sequence (9) yields an isotropic section and  $(A, \tilde{s})$  is defined to be the characteristic pair of  $L$ . It is well defined since for any two isotropic sections  $s_1, s_2$ , we have  $\text{Im}(s_1 - s_2) \subset \mathbf{b}(L)^0 = A^0$ , which is equivalent to  $\tilde{s}_1 = \tilde{s}_2$ .

Conversely, given a subbundle  $A \subset \mathcal{T}$  and any characteristic pair  $(A, \tilde{s})$ , set  $L_{s,A} = s(A) \oplus A^0$ . Evidently,  $L_{s,A}$  is a maximal isotropic subbundle of  $\mathcal{E}$  whose characteristic pair is  $(A, \tilde{s})$ . It is also clear that if  $\tilde{s}_1 = \tilde{s}_2$ ,  $L_{s_1,A} = L_{s_2,A}$ .

One may check that these two constructions are inverse to each other. ■

**Definition 3.4.** A *projective Lie algebroid* is a subbundle  $A \subset TM \oplus E$  which is a Lie algebroid  $(A, [\cdot, \cdot]_A, \rho_A)$  and the anchor  $\rho_A = \text{pr}_{TM}|_A$ .

**Example 3.5.** Let  $\mathcal{A} \longrightarrow N$  be a Lie algebroid over a smooth manifold  $N$  and  $\alpha$  its anchor. Let  $f : M \longrightarrow N$  be a smooth map and  $f^*\mathcal{A} \rightarrow M$  the pull back bundle along  $f$ . We denote the pull back Lie algebroid of  $\mathcal{A}$  over  $M$  by  $f^!\mathcal{A} = TM \oplus_{TN} \mathcal{A}$ , which is given by

$$TM \oplus_{TN} \mathcal{A} = \{(x, X) \in T_m M \oplus \mathcal{A}_{f(m)} | m \in M, \text{ and } f_*(x) = \alpha(X)\}.$$

Sections of  $TM \oplus_{TN} \mathcal{A}$  are of the form:

$$x \oplus \left( \sum u_i \otimes X_i \right), \quad x \in \mathfrak{X}(M), \quad u_i \in C^\infty(M), \quad X_i \in \Gamma(\mathcal{A}),$$

such that  $f_*(x(m)) = \sum u_i(m) \alpha(X_i(f(m)))$ . The anchor  $\alpha^!$  of the Lie algebroid  $f^!\mathcal{A}$  is the projection to the first summand. The Lie bracket can be *locally* expressed by

$$\begin{aligned} & [x \oplus \left( \sum u_i \otimes X_i \right), y \oplus \left( \sum v_j \otimes Y_j \right)] \\ &= [x, y] \oplus \left( \sum u_i v_j \otimes [X_i, Y_j] + \sum x(v_j) \otimes Y_j - \sum y(u_i) \otimes X_i \right). \end{aligned}$$

Thus the pull back Lie algebroid  $f^!\mathcal{A}$  of the Lie algebroid  $\mathcal{A}$  is a projective Lie algebroid in  $TM \oplus f^*\mathcal{A}$ .

**Example 3.6.** We suppose that the base manifold  $M$  is compact and let  $H \subset TM$  be an integrable distribution. It is well known that there is some vector bundle  $E$  such that the vector bundle  $F = H \oplus E$  is trivial. Suppose that  $\text{rank} F = n$  and  $\varepsilon_1, \dots, \varepsilon_n$  are everywhere linear independent sections of  $F$ , i.e. a frame of  $\Gamma(F)$ . Write  $\varepsilon_i = x_i + e_i$ , where  $x_i$  and  $e_i$  are sections of  $H$  and  $E$  respectively. It is clear that  $\Gamma(H) = \text{span}\{x_1, \dots, x_n\}$  and  $\Gamma(E) = \text{span}\{e_1, \dots, e_n\}$  (over  $C^\infty(M)$ ). Since  $H$  is an integrable distribution, there exist functions  $c_{i,j}^k \in C^\infty(M)$  such that  $[x_i, x_j] = c_{i,j}^k x_k$ . Now set  $[\varepsilon_i, \varepsilon_j] = c_{i,j}^k \varepsilon_k$ . It is easy to see that  $F$  is a projective Lie algebroid in  $TM \oplus E$ .

A Dirac structure  $L \subset \mathcal{E}$  is called **reducible** if  $\mathbf{b}(L)$  is a regular subbundle of  $\mathcal{T}$ . By Lemma 3.1, any Dirac structure is reducible if  $\text{rank}(E) \geq 2$ . As a main result of this paper, the following theorem describes the nature of reducible Dirac structures in the omni-Lie algebroid  $\mathcal{E}$ .

**Theorem 3.7.** *For any vector bundle  $E$ , there is a one-to-one correspondence between reducible Dirac structures  $L \subset \mathcal{E}$  and projective Lie algebroids  $A = \mathbf{b}(L) \subset \mathcal{T}$  such that  $A$  is the quotient Lie algebroid of  $L$ .*

**Proof.** Assume that  $L$  is a reducible Dirac structure and let  $A = \mathbf{b}(L) \subset \mathcal{T}$ . Then we have the following exact sequence:

$$0 \longrightarrow A^0 \xrightarrow{\mathbf{a}} L \xrightarrow{\mathbf{b}} A \longrightarrow 0. \quad (10)$$

By  $L$  being reducible,  $A$  is a regular subbundle,  $A^0$  as well. The anchor  $\alpha \circ \rho$  vanishes if restricted on  $A^0$ . Furthermore, by Lemma 2.6 and the fact that  $L$  is a Dirac structure,  $A^0$  is an ideal of  $L$ . So we have a quotient Lie algebroid structure  $(A, [\cdot, \cdot]_A, \rho_A)$ , where  $\rho_A$  is clearly the projection to  $TM$ . This proves that  $A$  is indeed a projective Lie algebroid.

Conversely, for the projective Lie algebroid  $(A, [\cdot, \cdot]_A, \rho_A)$ , define a subset  $L^A \subset \mathbf{b}^{-1}(A) \subset \mathcal{E}$  by:

$$\begin{aligned} L_m^A &\triangleq \{X \in \mathbf{b}^{-1}(A)_m \mid \text{for some } \tilde{X} \in \Gamma(\mathbf{b}^{-1}(A)) \text{ with } \tilde{X}_m = X, \text{ there holds} \\ &\quad \mathbf{b}\{\tilde{X}, Y\}_m = ([\mathbf{b}\tilde{X}, \mathbf{b}Y]_A)_m, \quad \forall Y \in \Gamma(\mathbf{b}^{-1}(A))\}. \end{aligned} \quad (11)$$

Note that by Equation (4), we have

$$\mathbf{b}\{f\tilde{X}, Y\}_m - ([f\mathbf{b}\tilde{X}, \mathbf{b}Y]_A)_m = f(\mathbf{b}\{\tilde{X}, Y\}_m - ([\mathbf{b}\tilde{X}, \mathbf{b}Y]_A)_m).$$

Hence the above definition does not depend on the choice of  $\tilde{X}$ .

To prove that  $L^A$  is the unique reducible Dirac structure such that the induced projective Lie algebroid is  $(A, [\cdot, \cdot]_A, \rho_A)$ , we need three steps as follows. Step 1 proves that  $L^A$  is a maximal isotropic subbundle such that  $\mathbf{b}(L^A) = A$ . Step 2 proves that  $L^A$  is closed under the bracket  $\{\cdot, \cdot\}$  and it follows that  $L^A$  is a reducible Dirac structure such that the induced projective Lie algebroid is  $(A, [\cdot, \cdot]_A, \rho_A)$ . The last step proves the uniqueness of such Dirac structures.

**Step 1.** We prove that  $L^A$  is a maximal isotropic subbundle. We will construct a maximal isotropic subbundle  $L_{s_\gamma, A}$  using a connection  $\gamma$  in the vector bundle  $E$  and prove that  $L_{s_\gamma, A} = L^A$ .

Recall that a connection in  $E$  is a bundle map  $\gamma: TM \rightarrow \mathfrak{D}E$  such that  $\alpha \circ \gamma = \mathbf{1}_{TM}$ . Associated with  $\gamma$  there is a back connection  $\omega: \mathfrak{D}E \rightarrow \mathfrak{gl}(E)$ , such that  $\mathfrak{i} \circ \omega + \gamma \circ \alpha = \mathbf{1}_{\mathfrak{D}E}$ . So we can define a bundle map  $\tilde{\gamma}: E \rightarrow \mathfrak{J}E$  by

$$\langle \tilde{\gamma}(e), \mathfrak{d} \rangle_E \triangleq \omega(\mathfrak{d})(e) = (\mathfrak{d} - \gamma \circ \alpha(\mathfrak{d}))(e), \quad \forall \mathfrak{d} \in \mathfrak{D}E \quad (12)$$

such that  $\mathfrak{p} \circ \tilde{\gamma} = \mathbf{1}_E$ . In turn, we get a map:

$$\gamma + \tilde{\gamma}: \mathcal{T} \rightarrow \mathcal{E} \quad \text{such that} \quad \mathbf{b} \circ (\gamma + \tilde{\gamma}) = \mathbf{1}_{\mathcal{T}}. \quad (13)$$



We still denote this map by  $\gamma$ . This does not make any confusion since it depends on what is put right after it.

Choose an arbitrary subbundle  $C \subset \mathcal{T}$ , such that  $\mathcal{T} = A \oplus C$ . Define a bundle map  $\Omega_\gamma : \mathcal{T} \wedge \mathcal{T} \rightarrow E$  by

$$\begin{aligned}\Omega_\gamma(a, b) &= [a, b]_A - \mathbf{b}\{\gamma(a), \gamma(b)\}, \quad \forall a, b \in \Gamma(A), \\ \Omega_\gamma(c, t) &= 0, \quad \forall c \in C, t \in \mathcal{T}.\end{aligned}$$

To see that  $\Omega_\gamma \in \text{Hom}(\wedge^2 \mathcal{T}, E)$ , first for any  $a = x + u$ ,  $b = y + v \in \Gamma(A)$ , where  $x, y \in \mathfrak{X}(M)$ ,  $u, v \in \Gamma(E)$ , we have

$$\begin{aligned}\mathbf{b}\{\gamma(x + u), \gamma(y + v)\} &= \mathbf{b}([\gamma(x), \gamma(y)]_{\mathfrak{D}} + \mathfrak{L}_{\gamma(x)}\gamma(v) - \mathfrak{L}_{\gamma(y)}\gamma(u) + \mathfrak{d}\langle \gamma(y), \gamma(u) \rangle_E) \\ &= [\alpha\gamma(x), \alpha\gamma(y)]_{\mathfrak{D}} + \gamma(x)(\mathbb{P}\gamma(v)) - \gamma(y)(\mathbb{P}\gamma(u)) \\ &= [x, y] + \gamma(x)v - \gamma(y)u,\end{aligned}$$

which implies that

$$\Omega_\gamma(x + u, y + v) = ([x + u, y + v]_A - [x, y]) - \gamma(x)(v) + \gamma(y)(u). \quad (14)$$

Thus we have  $\Omega_\gamma(x + u, y + v) \in \Gamma(E)$ . On the other hand, for any  $f \in C^\infty(M)$ , we have

$$\begin{aligned}\Omega_\gamma(x + u, f(y + v)) &= ([x + u, f(y + v)]_A - [x, fy]) - \gamma(x)(fv) + \gamma(fy)(u) \\ &= f\Omega_\gamma(x + u, y + v) + x(f)(y + v) - x(f)y - \alpha(\gamma(x))(f)v \\ &= f\Omega_\gamma(x + u, y + v).\end{aligned} \quad (15)$$

By (14) and (15), we obtain that  $\Omega_\gamma \in \text{Hom}(\wedge^2 \mathcal{T}, E)$ . We also denote the associated skew-symmetric map from  $\mathcal{T}$  to  $\text{Hom}(\mathcal{T}, E)$  by  $\Omega_\gamma$ .

Define an isotropic section  $s_\gamma : A \rightarrow \mathcal{E}$  by

$$s_\gamma(a) = \gamma(a) + \Omega_\gamma(a), \quad \forall a \in A.$$

In fact, for  $a = x + u$ ,  $b = y + v \in \Gamma(A)$ , we have

$$\begin{aligned}& (s_\gamma(x + u), s_\gamma(y + v))_E \\ &= (\gamma(x) + \gamma(u) + \Omega_\gamma(a), \gamma(y) + \gamma(v) + \Omega_\gamma(b))_E \\ &= \frac{1}{2}(\Omega_\gamma(y + v, x + u) + \Omega_\gamma(x + u, y + v) + \langle \gamma(x), \gamma(v) \rangle_E + \langle \gamma(y), \gamma(u) \rangle_E) = 0.\end{aligned}$$

By Proposition 3.3, we get a maximal isotropic subbundle  $L_{s_\gamma, A}$ :

$$L_{s_\gamma, A} = \gamma(A) + \Omega_\gamma(A) + A^0. \quad (16)$$

We can directly check that  $L_{s_\gamma, A}$  does not depend on the choice of the connection  $\gamma$  and the subbundle  $C$ . An alternate approach is to prove that  $L_{s_\gamma, A} = L^A$ , since  $L^A$  does not depend on  $s_\gamma$  and  $A$ .

Now we prove  $L_{s_\gamma, A} = L^A$ . Any  $X \in \Gamma(L_{s_\gamma, A})$  has the form  $X = \gamma(a) + \Omega_\gamma(a) + h$ , where  $a = x + u \in \Gamma(A)$  and  $h \in \Gamma(A^0)$ . For any  $Y = \mathfrak{d} + \mu \in \Gamma(\mathbf{b}^{-1}(A))$  satisfying

$\mathbf{b}(Y) = y + v \in \Gamma(A)$ , we have

$$\begin{aligned}
\mathbf{b}\{X, Y\} &= \mathbf{b}(\{\gamma(x) + \gamma(u), \mathfrak{d} + \mu\} + \{\Omega_\gamma(a) + h, Y\}) \\
&= \mathbf{b}([\gamma(x), \mathfrak{d}]_{\mathfrak{D}} + \mathfrak{L}_{\gamma(x)}\mu - \mathfrak{L}_{\mathfrak{d}}\gamma(u) + \mathfrak{d}\langle\gamma(u), \mathfrak{d}\rangle_E) + (\Omega_\gamma(a) + h)(\mathbf{b}(Y)) \\
&= [x, \alpha\mathfrak{d}] + \gamma(x)(v) - \mathfrak{d}(u) + \langle\gamma(u), \mathfrak{d}\rangle_E + \Omega_\gamma(x + u, y + v) \\
&= [x, y] + \gamma(x)v - \gamma(y)u + \Omega_\gamma(x + u, y + v) \\
&= [x + u, y + v]_A, \quad (\text{using (14)}) \\
&= [\mathbf{b}(X), \mathbf{b}(Y)]_A.
\end{aligned}$$

Thus,  $X \in \Gamma(L^A)$ . So we have  $L_{s_\gamma, A} \subset L^A$ . Since  $\mathbf{b}(L^A) \subset A$ , any  $X \in L^A$  can be written as  $X = X_0 + h$ , where  $X_0 \in L_{s_\gamma, A}$  and  $h \in \text{Hom}(\mathcal{T}, E)$ . Thus  $h = X - X_0 \in L^A \cap \text{Hom}(\mathcal{T}, E)$ .

For any  $k \in \text{Hom}(\mathcal{T}_m, E_m) = \text{Ker } \mathbf{b}$  and  $\tilde{k} \in \Gamma(\text{Hom}(\mathcal{T}, E))$  satisfying  $\tilde{k}(m) = k$ ,  $\forall Y \in \Gamma(\mathbf{b}^{-1}(A))$ , we have, by Equation (7)

$$\mathbf{b}\left\{\tilde{k}, Y\right\}_m - ([\mathbf{b}(\tilde{k}), \mathbf{b}(Y)]_A)_m = k(\mathbf{b}(Y)).$$

Thus  $k \in L_m^A \cap \text{Hom}(\mathcal{T}_m, E_m)$  if and only if  $k \in A_m^0$ , that is,

$$L^A \cap \text{Hom}(\mathcal{T}, E) = A^0. \quad (17)$$

So we have proved that  $L^A \subset L_{s_\gamma, A}$ . By maximality,  $L^A = L_{s_\gamma, A}$  and hence  $L^A$  is a maximal isotropic subbundle of  $\mathcal{E}$ .

**Step 2.** We prove that  $\Gamma(L^A)$  is closed under the bracket operation  $\{\cdot, \cdot\}$  and it follows that  $L^A = L_{s_\gamma, A}$  is a reducible Dirac structure.

For any  $X_1, X_2 \in \Gamma(L^A)$  and  $Y \in \Gamma(\mathbf{b}^{-1}(A))$ , we have  $\{X_1, X_2\} \in \Gamma(\mathbf{b}^{-1}(A))$  and  $\{X_i, Y\} \in \Gamma(\mathbf{b}^{-1}(A))$ . Moreover, we have

$$\begin{aligned}
\mathbf{b}\{\{X_1, X_2\}, Y\} &= \mathbf{b}\{X_1, \{X_2, Y\}\} - \mathbf{b}\{X_2, \{X_1, Y\}\} \\
&= [\mathbf{b}X_1, \mathbf{b}\{X_2, Y\}]_A - [\mathbf{b}X_2, \mathbf{b}\{X_1, Y\}]_A \\
&= [\mathbf{b}X_1, [\mathbf{b}X_2, Y]_A]_A - [\mathbf{b}X_2, [\mathbf{b}X_1, Y]_A]_A \\
&= [[\mathbf{b}X_1, \mathbf{b}X_2]_A, \mathbf{b}Y]_A \\
&= [\mathbf{b}\{X_1, X_2\}, \mathbf{b}Y]_A,
\end{aligned}$$

which implies that  $\{X_1, X_2\} \in \Gamma(L^A)$ . So  $L^A$  is a Dirac structure. In Step 1, we have proved that  $\mathbf{b}(L^A) = A$ , and in turn,  $L^A$  is a reducible Dirac structure. By definition, the induced projective Lie algebroid is exactly  $(A, [\cdot, \cdot]_A, \rho_A)$ .

**Step 3.** We prove the uniqueness of such Dirac structures.

Assume that  $L'$  is another reducible Dirac structure satisfying the same requirements. It suffices to prove that  $L' \subset L^A$ , since  $L^A$  is a maximal isotropic subbundle. For any  $X \in L'_m$  and  $\tilde{X} \in \Gamma(L')$  such that  $\tilde{X}_m = X$ , we prove that  $X \in L_m^A$ . In fact,  $\forall Y \in \Gamma(\mathbf{b}^{-1}(A))$ , we are able to find some  $Y' \in \Gamma(L')$  such that  $\mathbf{b}Y' = \mathbf{b}Y$ . So we can write  $Y = Y' + K$ , where  $K \in \Gamma(\text{Hom}(\mathcal{T}, E))$ . By Lemma 2.6,  $\{\tilde{X}, K\} \in \Gamma(\text{Hom}(\mathcal{T}, E))$ . Thus,

$$\mathbf{b}\{\tilde{X}, Y\} = \mathbf{b}\{\tilde{X}, Y'\} + \mathbf{b}\{\tilde{X}, K\} = [\mathbf{b}\tilde{X}, \mathbf{b}Y']_A = [\mathbf{b}\tilde{X}, \mathbf{b}Y]_A,$$

which implies that  $X \in L_m^A$ . So we have  $L' \subset L^A$ . The proof of Theorem 3.7 is thus completed. ■

The projective Lie algebroid  $\mathbf{b}(L)$  is called the **reduction** of the reducible Dirac structure  $L$ . The reducible Dirac structure  $L^A$  is called the **lift** of the projective Lie algebroid  $A$ .

## 4 Some Examples

Bellow we give some basic examples of Dirac structures in the omni-Lie algebroid.

**Example 4.1.** For a vector space  $V$ , our theorem claims a one-to-one correspondence between Dirac structures of the omni-Lie algebra  $\mathfrak{gl}(V) \oplus V$  and Lie algebra structures on subspaces of  $V$ . Thus Dirac structures characterize not only all Lie algebra structures on  $V$ , as pointed out by Weinstein [31], but also all Lie algebra structures on subspaces of  $V$ .

**Example 4.2.** Given a skew-symmetric bundle map  $\hat{\lambda} : \mathfrak{D}E \rightarrow \mathfrak{J}E$ , its graph

$$L^{\hat{\lambda}} = \left\{ \mathfrak{d} + \hat{\lambda}(\mathfrak{d}) \mid \forall \mathfrak{d} \in \mathfrak{D}E \right\} \subset \mathcal{E}$$

is clearly a maximal isotropic subbundle. Furthermore, we have  $\hat{\lambda}(\mathfrak{gl}(E)) \subset \text{Hom}(TM, E)$ , i.e.  $\mathfrak{p}\hat{\lambda}(\Phi) = 0$ . In fact,  $\forall \Phi \in \mathfrak{gl}(E)$ , we have

$$\left\langle \hat{\lambda}(\Phi), \mathbf{1}_E \right\rangle_E = \mathfrak{p}\hat{\lambda}(\Phi), \quad \left\langle \hat{\lambda}(\mathbf{1}_E), \Phi \right\rangle_E = \Phi \circ \mathfrak{p}\hat{\lambda}(\mathbf{1}_E).$$

Since  $\hat{\lambda}$  is skew-symmetric, we have  $\mathfrak{p}\hat{\lambda}(\Phi) = -\Phi \circ \mathfrak{p}\hat{\lambda}(\mathbf{1}_E)$ . If we take  $\Phi = \mathbf{1}_E$ , then  $\mathfrak{p}\hat{\lambda}(\mathbf{1}_E) = 0$ . Thus,  $\mathfrak{p}\hat{\lambda}(\Phi) = 0$ .

Let  $\lambda : TM \rightarrow E$  be the induced bundle map of  $\hat{\lambda}$ . Then we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{gl}(E) & \xrightarrow{\quad \mathfrak{i} \quad} & \mathfrak{D}E & \xrightarrow{\quad \alpha \quad} & TM \longrightarrow 0 \\ & & \downarrow -\lambda^* & & \downarrow \hat{\lambda} & & \downarrow \lambda \\ 0 & \longrightarrow & \text{Hom}(TM, E) & \xrightarrow{\quad \mathfrak{e} \quad} & \mathfrak{J}E & \xrightarrow{\quad \mathfrak{p} \quad} & E \longrightarrow 0. \end{array}$$

Here  $-\lambda^*$  is induced by  $\hat{\lambda}|_{\mathfrak{gl}(E)}$ , which is given by  $\Phi \mapsto -\Phi \circ \lambda$ . So we have the following exact sequence:

$$0 \longrightarrow \mathbf{G}_{-\lambda^*} \longrightarrow L^{\hat{\lambda}} \longrightarrow \mathbf{G}_{\lambda} \longrightarrow 0,$$

where  $\mathbf{G}_{\lambda} = \mathbf{b}(L^{\hat{\lambda}})$  is the graph of  $\lambda$  and  $\mathbf{G}_{-\lambda^*} = L^{\hat{\lambda}} \cap \text{Hom}(T, E)$  is the graph of  $-\lambda^*$ .

We claim that the following three statements are equivalent.

- 1)  $L^{\hat{\lambda}}$  is a Dirac structure.
- 2)  $\mathfrak{d}\hat{\lambda} = 0$ , regarding  $\hat{\lambda}$  as a map  $\mathfrak{D}E \wedge \mathfrak{D}E \rightarrow E$  in the obvious sense:

$$\hat{\lambda}(\mathfrak{d}, \mathfrak{r}) = \left\langle \hat{\lambda}(\mathfrak{d}), \mathfrak{r} \right\rangle_E, \quad \forall \mathfrak{d}, \mathfrak{r} \in \mathfrak{D}E.$$

- 3)  $\hat{\lambda} = -\mathfrak{d}(\lambda \circ \alpha)$ .

In fact,  $1) \iff 2)$  is merely some calculations.  $3) \implies 2)$  is trivial. To see the reverse, notice that  $\forall \mathfrak{r}, \mathfrak{s} \in \Gamma(\mathfrak{D}E)$ ,

$$\begin{aligned} \mathfrak{d}\hat{\lambda}(\mathbf{1}_E, \mathfrak{r}, \mathfrak{s}) &= \left\langle \hat{\lambda}(\mathfrak{r}), \mathfrak{s} \right\rangle_E - \mathfrak{r} \left\langle \hat{\lambda}(\mathbf{1}_E), \mathfrak{s} \right\rangle_E + \mathfrak{s} \left\langle \hat{\lambda}(\mathbf{1}_E), \mathfrak{r} \right\rangle_E - \left\langle \hat{\lambda}[\mathfrak{r}, \mathfrak{s}], \mathbf{1}_E \right\rangle \\ &= \hat{\lambda}(\mathfrak{r}, \mathfrak{s}) + \mathfrak{r}(\lambda \circ \alpha(\mathfrak{s})) - \mathfrak{s}(\lambda \circ \alpha(\mathfrak{r})) - (\lambda \circ \alpha)[\mathfrak{r}, \mathfrak{s}]_{\mathfrak{D}}, \end{aligned}$$

which implies that  $2) \implies 3)$ .

Thus, any Dirac structure of the type  $L^{\hat{\lambda}}$  is a reducible Dirac structure and totally determined by

$$\mathbf{b}(L^{\hat{\lambda}}) = \mathbf{G}_{\lambda} \subset \mathcal{T},$$

which is isomorphic to  $TM$  and equipped with the induced Lie algebroid structure.

**Example 4.3.** (See [6]) For a skew-symmetric bundle map  $\pi : \mathfrak{J}E \rightarrow \mathfrak{D}E$ , its graph

$$L_{\pi} = \{\pi(\mu) + \mu \mid \mu \in \mathfrak{J}E\} \subset \mathcal{E}$$

is clearly a maximal isotropic subbundle of  $\mathcal{E}$ . It can be proved that  $L_{\pi}$  is a Dirac structure if and only if the following equation holds for all  $\mu, \nu \in \Gamma(\mathfrak{J}E)$ ,

$$\pi[\mu, \nu]_{\pi} = [\pi(\mu), \pi(\nu)]_{\mathfrak{D}},$$

where the  $\pi$ -bracket  $[\cdot, \cdot]_{\pi}$  on  $\Gamma(\mathfrak{J}E)$  is given by:

$$[\mu, \nu]_{\pi} \triangleq \mathfrak{L}_{\pi(\mu)}\nu - \mathfrak{L}_{\pi(\nu)}\mu - \mathfrak{d} \circ \pi(\mu \wedge \nu). \quad (18)$$

To see what  $\pi$  encodes, we need to consider the following two situations:

- $\text{rank}(E) \geq 2$ . In this case, in [6], we proved that such Dirac structures are in one-to-one correspondence with Lie algebroid structures on  $E$ . Let us see how Theorem 3.7 recovers this result. On one hand, there is an obvious one-to-one correspondence between Lie algebroid structures  $(E, [\cdot, \cdot]_E, \rho_E)$  and projective Lie algebroids  $\mathbf{G}_{\rho_E}$  which are the graphs of  $\rho_E : E \rightarrow TM$ . On the other hand, by Lemma 3.1, any Dirac structure is reducible. Especially, for any Dirac structure  $L_{\pi} \subset \mathcal{E}$ ,  $\mathbf{b}(L_{\pi})$  should be a projective Lie algebroid. However,  $\mathbf{b}(L_{\pi})$  is also a graph and hence there is an induced Lie algebroid structure on  $E$ . So we conclude that Lie algebroid structures on  $E$  are in one-to-one correspondence with Dirac structures of the type  $L_{\pi}$ .

- $\text{rank}(E) = 1$ . For any reducible Dirac structure  $L_{\pi} \subset \mathcal{E}$ ,  $\mathbf{b}(L_{\pi})$  is a projective Lie algebroid. But in general, it may not be a graph and so there is no induced Lie algebroid structure on  $E$ . However, there is always a local Lie algebra structure on  $E$  associated with the Dirac structure  $L_{\pi}$  (not necessarily reducible) as proved in [6].

**Example 4.4.** Consider the case that  $A \subset \mathcal{T}$  is an arbitrary line bundle, which is naturally a projective Lie algebroid. In fact, for any neighborhood  $\mathcal{U} \subset M$  such that  $A|_{\mathcal{U}}$  is trivial, i.e. there is a nowhere singular section  $a = x + u$ , the Lie bracket of  $\Gamma(A|_{\mathcal{U}})$  is given by:

$$[fa, ga]_A = (fx(g) - gx(f))a, \quad \forall f, g \in C^{\infty}(\mathcal{U}).$$

It is easy to check that this bracket is well defined.

The lifted Dirac structure declared by Theorem 3.7 can be constructed by Equation (16). Just take any connection  $\gamma$ . Since  $A$  is a line bundle, we have  $\Omega_{\gamma}(A) \subset A^0$ . The lifted Dirac structure is given by  $L^A = L_{s_{\gamma}, A} = \gamma(A) \oplus A^0$ .

**Example 4.5.** Assume that  $F \subset E$  is a vector subbundle and  $(F, [\cdot, \cdot], \rho_F)$  is a Lie algebroid. Then  $\mathbf{G}_{\rho_F}$ , the graph of  $\rho_F$  is a projective Lie algebroid. Now we construct the lifted Dirac structure. Evidently, we have

$$\mathbf{G}_{\rho_F}^0 = \{\Phi + \eta \in \text{Hom}(\mathcal{T}, E) \mid (\eta \circ \rho_F + \Phi)|_F = 0\}.$$

Let  $L_1 \subset \mathcal{E}$  be the subset generated by elements of the form  $\mathfrak{d}_m^v + [v]_m$ , where  $m \in M$ ,  $v \in \Gamma(F)$ ,  $\mathfrak{d}_m^v \in (\mathfrak{D}E)_m$  and they satisfy

$$\mathfrak{d}_m^v(u) = ([v, u]_F)_m, \quad \forall u \in \Gamma(F).$$

Let  $L = L_1 + \mathbf{G}_{\rho_F}^0$ . Accordingly, we get an exact sequence:

$$0 \longrightarrow \mathbf{G}_{\rho_F}^0 \longrightarrow L \xrightarrow{\mathbf{b}} \mathbf{G}_{\rho_F} \longrightarrow 0.$$

It is clear that  $L$  is maximal isotropic. Moreover, for all  $u, v \in \Gamma(F)$ , we have

$$[\mathfrak{d}^u, \mathfrak{d}^v]_{\mathfrak{D}E} = \mathfrak{d}^{[u, v]_F}$$

and it follows that  $\Gamma(L)$  is closed under the bracket  $\{\cdot, \cdot\}$ . Hence  $L$  is the lifted Dirac structure.

**Example 4.6.** We consider a projective Lie algebroid  $A$  which is *transitive*, i.e.  $\rho_A(A) = \text{pr}_{TM}(A) = TM$ . One can construct a map  $\vartheta : TM \rightarrow E$  such that  $A = \mathbf{G}_{\vartheta} \oplus E_0$ , where  $\mathbf{G}_{\vartheta}$  is the graph of  $\vartheta$  and  $E_0$  is a subbundle of  $E$ . In this case,  $E_0$  must be a Lie algebra bundle and we denote its Lie bracket by  $[\cdot, \cdot]^0$ .

For any vector field  $x \in \mathfrak{X}(M)$ , write  $\bar{x} = x + \vartheta(x) \in \Gamma(A)$ . There is a suitable connection  $\gamma : TM \rightarrow \mathfrak{D}E_0$  such that

$$[\bar{x}, u]_A = \gamma(x)u, \quad \forall x \in \mathfrak{X}(M), u \in \Gamma(E_0).$$

Define  $R : \wedge^2 TM \rightarrow E_0$  by

$$R(x, y) = [\bar{x}, \bar{y}]_A - \overline{[x, y]}, \quad \forall x, y \in \mathfrak{X}(M).$$

So the Lie bracket of  $\Gamma(A)$  can be written as

$$[\bar{x} + u, \bar{y} + v]_A = \overline{[x, y]} + R(x, y) + \gamma(x)v - \gamma(y)u + [u, v]^0, \quad \forall \bar{x} + u, \bar{y} + v \in \Gamma(A).$$

Under the structure defined by the given data  $(\gamma, R)$ ,  $A = \mathbf{G}_{\vartheta} \oplus E_0$  is a Lie algebroid if and only if  $\forall x, y, z \in \mathfrak{X}(M)$ ,  $u, v \in \Gamma(E_0)$ , the following compatibility conditions hold

$$\begin{aligned} [\gamma(x)u, v]^0 + [u, \gamma(x)v]^0 &= \gamma(x)[u, v]^0, \\ [\gamma(x), \gamma(y)]_{\mathfrak{D}} - \gamma[x, y] &= \text{ad}_{R(x, y)}^{E_0}, \\ R([x, y], z) - \gamma(x)R(y, z) + c.p. &= 0. \end{aligned}$$

We extend the connection  $\gamma$  in the vector bundle  $E_0$  to a connection  $\tilde{\gamma}$  in the vector bundle  $E$ . By (14), we have

$$\Omega_{\tilde{\gamma}}(\bar{x} + u, \bar{y} + v) = R(x, y) + [u, v]^0 - (d\tilde{\gamma}\vartheta)(x, y),$$

where

$$(d\tilde{\gamma}\vartheta)(x, y) = \tilde{\gamma}(x)\vartheta(y) - \tilde{\gamma}(y)\vartheta(x) - \vartheta[x, y].$$

The lifted Dirac structure, given by Theorem 3.7, can be expressed by (16).

In particular, if  $A = \mathcal{T} = TM \oplus E$  (so that we may take  $\vartheta = 0$ ), then

$$L_{s_{\gamma}, \mathcal{T}} = \left\{ \gamma(x + u) + i_x R + \text{ad}_u^E \mid \forall x + u \in TM \oplus E \right\}.$$

We note that the above construction of projective Lie algebroids includes a standard type of Lie algebroids, known as a *semi-direct-sum*. If  $E$  is a vector bundle over  $M$  which admits a **flat** connection  $\nabla : TM \rightarrow \mathfrak{D}E$ , then the direct sum  $TM \oplus E$  has a Lie algebroid structure over  $M$ , for which the anchor is the projection to  $TM$  and the Lie bracket is given by:

$$[x + u, y + v] \triangleq [x, y] + \nabla_x v - \nabla_y u, \quad \forall x + u, y + v \in \Gamma(TM \oplus E).$$

## 5 The Normalizer of Dirac Structures

In this and the next section, we always assume that Lie algebroids under consideration are not zero. For a Lie algebroid  $A$ , call  $\text{Der}(A)$ , the set of Lie derivations of  $A$ :

$$\text{Der}(A) = \{ \delta \in \Gamma(\mathfrak{D}A) \mid \delta[a_1, a_2]_A = [\delta a_1, a_2]_A + [a_1, \delta a_2]_A, \quad \forall a_1, a_2 \in \Gamma(A), \}$$

the derivation algebra of  $A$ .

**Definition 5.1.** *The normalizer  $N_C$  of a subbundle  $C$  of the omni-Lie algebroid  $\mathcal{E} = \mathfrak{D}E \oplus \mathfrak{J}E$  is composed of all the sections of  $\mathcal{E}$  that preserve  $\Gamma(C)$  from the left side, that is,*

$$N_C = \{ X \in \Gamma(\mathcal{E}) \mid \{X, Y\} \in \Gamma(C), \quad \forall Y \in \Gamma(C) \}. \quad (19)$$

It is easy to see that the normalizer  $N_C$  of  $C$  is a Leibniz subalgebra<sup>2</sup> of  $\Gamma(\mathcal{E})$ .

For any  $X \in \Gamma(\mathcal{E})$ , we introduce the *falling* operator

$$(\cdot)_{\bullet} : \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathfrak{D}\mathcal{T}),$$

which is defined by

$$X_{\bullet}(t) \triangleq \mathbf{b}\{X, Y\}, \quad \forall t \in \Gamma(\mathcal{T}), \quad (20)$$

where  $Y \in \Gamma(\mathcal{E})$  satisfying  $\mathbf{b}(Y) = t$ . By Lemma 2.6, this is well defined and if  $h \in \Gamma(\text{Hom}(\mathcal{T}, E))$ ,  $h_{\bullet} = h$ .

In this section, we study the normalizer  $N_L$  of a Dirac structure  $L$ . Using the falling operator defined above, we establish the relation between the normalizer  $N_L$  of a reducible Dirac structure  $L$  and the derivation algebra  $\text{Der}(\mathbf{b}(L))$  of the projective Lie algebroid  $\mathbf{b}(L)$ .

**Proposition 5.2.** *The falling operator  $(\cdot)_{\bullet}$  is a morphism of Leibniz algebras. Furthermore,  $\forall X \in \Gamma(\mathcal{E}), t \in \Gamma(\mathcal{T})$ , we have*

$$\text{pr}_{TM}(X_{\bullet}(t)) = [\alpha \circ \rho(X), \text{pr}_{TM}(t)] = [\alpha(X_{\bullet}), \text{pr}_{TM}(t)]. \quad (21)$$

*Conversely, given any  $\delta \in \Gamma(\mathfrak{D}\mathcal{T})$  satisfying Equation (21), there exists an  $X_{\delta} \in \Gamma(\mathcal{E})$  such that  $X_{\delta\bullet} = \delta$ .*

---

<sup>2</sup>Analogously, we may define  $N'_C$  to be the set of sections of  $\mathcal{E}$  that preserve  $C$  from the right side. But it is not a Leibniz subalgebra.

**Proof.** For all  $X, Y \in \Gamma(\mathcal{E})$ ,  $t \in \Gamma(\mathcal{T})$  and  $Z \in \Gamma(\mathcal{E})$  satisfying  $\mathbf{b}(Z) = t$ , we have

$$\begin{aligned} \{X, Y\}_\bullet(t) &= \mathbf{b}\{\{X, Y\}, Z\} = \mathbf{b}(\{X, \{Y, Z\}\} - \{Y, \{X, Z\}\}) \\ &= X_\bullet \mathbf{b}\{Y, Z\} - Y_\bullet \mathbf{b}\{X, Z\} \\ &= X_\bullet \circ Y_\bullet(t) - Y_\bullet \circ X_\bullet(t) \\ &= [X_\bullet, Y_\bullet]_{\mathfrak{D}}(t), \end{aligned}$$

which implies that the falling operator  $(\cdot)_\bullet$  is a morphism of Leibniz algebras.

Given  $X = \mathfrak{d} + \mu$  and  $Z = \mathfrak{r} + \nu$  such that  $pr_{TM}(t) = \alpha(\mathfrak{r})$ , we have

$$\begin{aligned} pr_{TM}(X_\bullet(t)) &= pr_{TM}\mathbf{b}\{X, Z\} \\ &= [\alpha(\mathfrak{d}), \alpha(\mathfrak{r})] = [\alpha \circ \rho(X), pr_{TM}(t)] \\ &= [\alpha(X_\bullet), pr_{TM}(t)], \end{aligned}$$

which implies Equation (21).

Suppose that  $\delta \in \Gamma(\mathfrak{DT})$  satisfies Equation (21). Write  $x = \alpha(\delta)$  and define  $\chi = pr_E \circ \delta$ . One has

$$\chi(ft) = x(f)pr_E(t) + f\chi(t), \quad \forall f \in C^\infty(M).$$

Therefore,  $\chi|_{\mathfrak{X}(M)}$  is  $C^\infty(M)$ -linear and there is an associated  $X_M \in \Gamma(\text{Hom}(TM, E))$ . Moreover,  $\chi|_{\Gamma(E)}$  is a derivation and there is an associated  $X_E \in \Gamma(\mathfrak{D}E)$  such that  $\alpha(X_E) = x$ . In turn, the operation of  $\delta$  can be expressed as

$$\delta(y + v) = [x, y] + X_E(v) + X_M(y), \quad \forall y + v \in \Gamma(\mathcal{T}).$$

Let  $X_\delta = X_E + X_M \in \Gamma(\mathfrak{D}E) \oplus \Gamma(\text{Hom}(TM, E)) \subset \Gamma(\mathcal{E})$ . We claim that  $X_{\delta_\bullet} = \delta$ . In fact, for any  $y + v \in \Gamma(\mathcal{T}) = \Gamma(TM) \oplus \Gamma(E)$  and  $Y = \mathfrak{r} + \nu \in \Gamma(\mathcal{E})$  satisfying  $\alpha(\mathfrak{r}) = y$  and  $\mathfrak{p}(\nu) = v$ , we have

$$\begin{aligned} X_{\delta_\bullet}(y + v) &= \mathbf{b}\{X, Y\} = \mathbf{b}([X_E, \mathfrak{r}]_{\mathfrak{D}} + \mathfrak{L}_{X_E}\nu - \mathfrak{L}_{\mathfrak{r}}X_M + \mathfrak{d}(X_M(y))) \\ &= [x, y] + X_E(v) + X_M(y) = \delta(y + v). \quad \blacksquare \end{aligned}$$

Let  $A \subset \mathcal{T}$  be a projective Lie algebroid and  $\text{Inn}(A)$  the set of inner derivations, which consists of operators  $[a, \cdot]_A$ , where  $a \in \Gamma(A)$ . Denote the set of external derivations by  $\text{Ext}(A)$ , i.e.

$$\text{Ext}(A) = \text{Der}(A)/\text{Inn}(A). \quad (22)$$

By Theorem 3.7, there is a unique lifted Dirac structure  $L^A$  such that  $A$  is the quotient Lie algebroid of  $L^A$ . Concerning the relation between the normalizer  $N_{L^A}$  and the derivation algebra  $\text{Der}(A)$ , we have

**Theorem 5.3.** *If  $X \in N_{L^A}$ , then  $X_\bullet|_A \in \text{Der}(A)$ . Conversely, for any  $\delta \in \text{Der}(A)$ , there exists an  $X_\delta \in N_{L^A}$ , such that  $(X_\delta)_\bullet|_A = \delta$ . Moreover, we have the following commutative diagram where the two rows are exact sequences:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(A^0) & \xrightarrow{i} & \Gamma(L^A) & \xrightarrow{(\cdot)_\bullet|_A} & \text{Inn}(A) \longrightarrow 0 \\ & & \downarrow i & & \downarrow i & & \downarrow i \\ 0 & \longrightarrow & \Gamma(A^0) \oplus \Gamma(E) & \xrightarrow{\kappa} & N_{L^A} & \xrightarrow{(\cdot)_\bullet|_A} & \text{Der}(A) \longrightarrow 0. \end{array}$$

Here  $i$  is the inclusion. The map  $\kappa$  is defined by  $\kappa(\phi + u) = \phi + \mathfrak{d}u$ ,  $\forall \phi \in \Gamma(A^0)$ ,  $u \in \Gamma(E)$ . In particular,  $X_\bullet|_A \in \text{Inn}A$  if and only if  $X = l + \mathfrak{d}u$ , for some  $l \in \Gamma(L^A)$ ,  $u \in \Gamma(E)$ .

**Proof.** If  $X \in N_{L^A}$ , then for any  $a_1, a_2 \in \Gamma(A)$ , we can find  $l_1, l_2 \in \Gamma(L^A)$  such that  $\mathfrak{b}(l_i) = a_i$ . Hence

$$\begin{aligned} X_\bullet[a_1, a_2]_A &= \mathfrak{b}\{X, \{l_1, l_2\}\} = \mathfrak{b}\{\{X, l_1\}, l_2\} + \mathfrak{b}\{l_1, \{X, l_2\}\} \\ &= [\mathfrak{b}\{X, l_1\}, a_2]_A + [a_1, \mathfrak{b}\{X, l_2\}]_A \\ &= [X_\bullet a_1, a_2]_A + [a_1, X_\bullet a_2]_A, \end{aligned}$$

which implies that  $X_\bullet|_A \in \text{Der}(A)$ .

Conversely, given any  $\delta \in \text{Der}(A)$ , set  $x = \alpha(\delta) \in \mathfrak{X}(M)$  and find an extension  $\tilde{\delta} \in \text{Der}(\mathcal{T})$  of  $\delta$ , that is,  $\alpha(\tilde{\delta}) = x$  and  $\tilde{\delta}|_{\Gamma(A)} = \delta$ . Since the elements of  $\text{Der}(A)$  satisfy (21) and by Proposition 5.2, there is an  $X_{\tilde{\delta}} = X_E + X_M$  such that  $X_{\tilde{\delta}\bullet} = \tilde{\delta}$ , i.e.  $X_{\tilde{\delta}\bullet}|_A = \delta$ .

Next we prove  $X_{\tilde{\delta}} \in N_{L^A}$ . For all  $l \in \Gamma(L^A)$ , it is evident that  $\{X_{\tilde{\delta}}, l\} \in \Gamma(\mathfrak{b}^{-1}(A))$ . Furthermore,  $\forall Y \in \Gamma(\mathfrak{b}^{-1}(A))$ , we have

$$\begin{aligned} \mathfrak{b}\{\{X_{\tilde{\delta}}, l\}, Y\} &= \mathfrak{b}\{X_{\tilde{\delta}}, \{l, Y\}\} - \mathfrak{b}\left\{l, \{X_{\tilde{\delta}}, Y\}\right\} \\ &= X_{\tilde{\delta}\bullet}[\mathfrak{b}l, \mathfrak{b}Y]_A - [\mathfrak{b}l, X_{\tilde{\delta}\bullet}(\mathfrak{b}Y)]_A \\ &= [X_{\tilde{\delta}\bullet}(\mathfrak{b}l), \mathfrak{b}(Y)]_A = [\mathfrak{b}\{X_{\tilde{\delta}}, l\}, \mathfrak{b}Y]_A, \end{aligned}$$

which implies that  $X_{\tilde{\delta}} \in N_{L^A}$ .

For an  $X \in N_{L^A}$  satisfying  $X_\bullet(\Gamma(A)) = 0$ , it is easy to see that  $\alpha(X_\bullet) = 0$ , i.e.  $\alpha \circ \rho(X) = 0$ . So we are able to write

$$X = \Phi + \mathfrak{y} + \mathfrak{d}u, \quad \text{where } \Phi \in \Gamma(\mathfrak{gl}(E)), \mathfrak{y} \in \Gamma(\text{Hom}(TM, E)), u \in \Gamma(E).$$

Clearly,  $\{\mathfrak{d}u, \cdot\} = 0$ . By Lemma 2.6, we have  $\Phi + \mathfrak{y} \in \Gamma(A^0)$ , which implies that  $\ker(X_\bullet|_A) = \Gamma(A^0) \oplus \Gamma(E)$ . The remaining statements of the theorem are easy to be checked and we omit the details. ■

**Example 5.4.** For a reducible Dirac structure  $L_\pi$  given in Example 4.3, we consider its normalizer. For  $u \in \Gamma(E)$ , since we have  $\{\mathfrak{d}u, L_\pi\} = 0$ , it suffices to consider elements of the form  $\mathfrak{d} + \mathfrak{y} \in \Gamma(\mathcal{E})$ , where  $\mathfrak{y} \in \Gamma(\text{Hom}(TM, E))$ . Rewrite  $\mathfrak{d} + \mathfrak{y} = \mathfrak{d} - \pi(\mathfrak{y}) + \pi(\mathfrak{y}) + \mathfrak{y}$  where  $\pi(\mathfrak{y}) + \mathfrak{y} \in L_\pi$ . We have

$$\mathfrak{d} + \mathfrak{y} \in N_{L_\pi} \iff \mathfrak{d} - \pi(\mathfrak{y}) \in N_{L_\pi} \iff \mathfrak{L}_{\mathfrak{d} - \pi(\mathfrak{y})} \circ \pi = \pi \circ \mathfrak{L}_{\mathfrak{d} - \pi(\mathfrak{y})} \iff d(\mathfrak{d} - \pi(\mathfrak{y})) = 0.$$

Here the coboundary operator  $d$  is associated with cochain complex  $\Gamma(\text{Hom}(\wedge^\bullet \mathfrak{J}E, E))$  and the representation  $\pi : \mathfrak{J}E \rightarrow \mathfrak{D}E$  (known as the adjoint representation [7]). From this we get the following exact sequence:

$$0 \rightarrow \Gamma(\text{Hom}(TM, E)) \oplus \Gamma(E) \xrightarrow{\kappa} N_{L_\pi} \xrightarrow{p} B(\mathfrak{J}E, E) \cap \mathfrak{D}E \rightarrow 0,$$

where  $B(\mathfrak{J}E, E)$  is the set of 1-cocycles and the maps  $\kappa, p$  are given by

$$\kappa(\mathfrak{y} + u) = \pi(\mathfrak{y}) + \mathfrak{y} + \mathfrak{d}u, \quad p(\mathfrak{d} + \mathfrak{y} + [u]) = \mathfrak{d} - \pi(\mathfrak{y}),$$

where  $\mathfrak{y} \in \Gamma(\text{Hom}(TM, E))$ ,  $\mathfrak{d} \in \Gamma(\mathfrak{D}E)$ ,  $u \in \Gamma(E)$ .



## 6 Cohomology of Dirac Structures

By Proposition 2.5, for any Dirac structure  $L \subset \mathcal{E}$ , there is a representation  $\rho_L$  on  $E$ . Let  $d_L : \Gamma(\text{Hom}(\wedge^\bullet L, E)) \rightarrow \Gamma(\text{Hom}(\wedge^{\bullet+1} L, E))$  be the associated coboundary operator. In this section, we study the cohomology group  $H^\bullet(L, \rho_L)$  and explore the relation between  $N_L$  and  $H^1(L, \rho_L)$ . We also study the deformation of a Dirac structure, which is related with  $H^2(L, \rho_L)$ .

Let  $L \subset \mathcal{E}$  be a Dirac structure. For any  $X \in \Gamma(\mathcal{E})$ ,  $\omega_X = (X, \cdot)_E : L \rightarrow E$  naturally defines a 1-cochain. We first prove the following fact.

**Proposition 6.1.**  $X \in N_L \iff d_L \omega_X = 0$ .

**Proof.** For any  $l_1, l_2 \in \Gamma(L)$ , we have

$$\begin{aligned} d_L \omega_X(l_1, l_2) &= \rho_L(l_1) \omega_X(l_2) - \rho_L(l_2) \omega_X(l_1) - \omega_X(\{l_1, l_2\}) \\ &= \rho_L(l_1) (X, l_2)_E - \rho_L(l_2) (X, l_1)_E - (X, \{l_1, l_2\})_E \\ &= (\{l_1, X\}, l_2)_E - (\{l_2, X\}, l_1)_E - (X, \{l_2, l_1\})_E \\ &= -(\{X, l_1\}, l_2)_E + (\{X, l_2\}, l_1)_E - (X, \{l_2, l_1\})_E \\ &\quad + (2\text{d}(l_1, X)_E, l_2)_E - (2\text{d}(l_2, X)_E, l_1)_E \\ &= -2(\{X, l_1\}, l_2)_E - (X, \{l_2, l_1\})_E - \rho_L(l_1) (X, l_2)_E + \rho_L(l_2) (X, l_1)_E \\ &= -2(\{X, l_1\}, l_2)_E - d_L \omega_X(l_1, l_2). \end{aligned}$$

Therefore,

$$d_L \omega_X(l_1, l_2) = -(\{X, l_1\}, l_2)_E. \quad (23)$$

Since  $L^\perp = L$ , the above equality implies that  $X \in N_L \iff d_L \omega_X = 0$ . ■

**Proposition 6.2.** Let  $A \subset \mathcal{T}$  be a projective Lie algebroid and  $L^A$  the lifted Dirac structure, for any  $X \in N_{L^A}$ ,  $\omega_X$  is a coboundary if and only if  $X_\bullet \in \text{Inn} A$

**Proof.** By definition,  $\omega_X = d_{L^A} u$ , for some  $u \in \Gamma(E)$ , if and only if

$$(X - 2\text{d}u, L^A)_E = 0 \iff X = 2\text{d}u + l, \quad \text{for some } l \in \Gamma(L^A).$$

So the conclusion follows directly by Theorem 5.3. ■

**Corollary 6.3.** With the above notations, there is a natural inclusion

$$i : \text{Ext}(A) \rightarrow H^1(L^A, \rho_{L^A}), \quad i(\delta) = \omega_{X_\delta}, \quad \forall \delta \in \text{Der}(A).$$

where  $\text{Ext}(A)$  is defined by (22) and  $X_\delta$  is given in Theorem 5.3.

**Proof.** By Theorem 5.3, for any  $\delta \in \text{Der}(A)$ , there is an  $X_\delta \in N_{L^A}$  such that  $X_{\delta_\bullet}|_A = \delta$ . By Proposition 6.1,  $i(\delta) = \omega_{X_\delta}$  is closed.

To see that  $i$  is well defined, we note that  $\omega_{X_\delta}$  does not depend on the choice of  $X_\delta$  (by Theorem 5.3). And if  $\delta \in \text{Inn}(A)$ , then by Theorem 5.3 again,  $X_\delta = l + 2\text{d}u$ , where  $l \in \Gamma(L^A)$  and  $u \in \Gamma(E)$ . Therefore,  $\omega_{X_\delta} = d_{L^A} u$  is exact.

Finally, the previous proposition implies that  $i$  is injective. ■

Suppose that  $E$  and  $E^*$  are both Lie algebroids, respectively, with anchors  $\alpha$  and  $\alpha^*$ . Let  $d_* : \Gamma(\wedge^\bullet E) \rightarrow \Gamma(\wedge^{\bullet+1} E)$  be the Lie algebroid coboundary operator associated with the Lie

algebroid structure on  $E^*$ . So we have  $d_*^2 = 0$ . By definition,  $(E, E^*)$  is a Lie bialgebroid if the following equality holds:

$$d_*[u, v] = [d_*u, v] + [u, d_*v], \quad \forall u, v \in \Gamma(E). \quad (24)$$

(For more details about Lie bialgebroids, see [21] and [20]). The operator  $d_* : \Gamma(E) \rightarrow \Gamma(\wedge^2 E)$  can be lifted to a bundle map  $\hat{d}_* : \mathfrak{J}E \rightarrow \wedge^2 E$ , defined by

$$\hat{d}_*(\mathfrak{d}u) \triangleq d_*u, \quad \hat{d}_*(\mathfrak{d}f \otimes u) \triangleq d_*f \wedge u, \quad \forall u \in \Gamma(E), f \in C^\infty(M). \quad (25)$$

In [6], we proved that a Lie algebroid structure on  $E$  can be lifted to a bundle map  $\pi : \mathfrak{J}E \rightarrow \mathfrak{D}E$ , which is also a representation of the jet Lie algebroid  $(\mathfrak{J}E, [\cdot, \cdot]_\pi, \alpha \circ \pi)$  on  $E$ , where  $\pi$  is given by  $\pi(\mathfrak{d}u)(v) = [u, v]$  (known as the adjoint representation of a Lie algebroid) and the Lie bracket  $[\cdot, \cdot]_\pi$  is given by (18). So we have an induced tensor representation  $\tilde{\pi}$  of  $\mathfrak{J}E$  on  $\wedge^2 E$  given by

$$\tilde{\pi}(\mathfrak{d}u)(\mathcal{W}) = [u, \mathcal{W}], \quad \tilde{\pi}(\mathfrak{d}f \otimes u)(\mathcal{W}) = [\mathcal{W}, f] \wedge u, \quad \forall \mathcal{W} \in \Gamma(\wedge^2 E).$$

**Proposition 6.4.**

- 1) The pair  $(E, E^*)$  is a Lie bialgebroid if and only if  $\hat{d}_*$  is a 1-cocycle.
- 2) The pair  $(E, E^*)$  is a coboundary Lie bialgebroid (i.e.  $d_* = [\tau, \cdot]$ , for some  $\tau \in \Gamma(\wedge^2 E)$ ) if and only if  $\hat{d}_*$  is a coboundary.

**Proof.** For all  $u, v \in \Gamma(E)$  and  $f, g \in C^\infty(M)$ , we have the following three formulas which are given in [6]:

$$\begin{aligned} [\mathfrak{d}u, \mathfrak{d}v]_\pi &= \mathfrak{d}[u, v], \\ [\mathfrak{d}u, \mathfrak{d}f \otimes v]_\pi &= \mathfrak{d}\rho(u)(f) \otimes v + \mathfrak{d}f \otimes [u, v], \\ [\mathfrak{d}f \otimes u, \mathfrak{d}g \otimes v]_\pi &= \rho(u)(g)(\mathfrak{d}f \otimes v) - \rho(v)(f)(\mathfrak{d}g \otimes u). \end{aligned}$$

Denote the coboundary operator associated with the representation  $\tilde{\pi}$  by  $\mathcal{D}$ . We have

$$\begin{aligned} \mathcal{D}(\hat{d}_*)(\mathfrak{d}u, \mathfrak{d}v) &= \tilde{\pi}(\mathfrak{d}u)\hat{d}_*(\mathfrak{d}v) - \tilde{\pi}(\mathfrak{d}v)\hat{d}_*(\mathfrak{d}u) - \hat{d}_*([\mathfrak{d}u, \mathfrak{d}v]_\pi) \\ &= [u, d_*v] - [v, d_*u] - d_*[u, v], \\ \mathcal{D}(\hat{d}_*)(\mathfrak{d}u, \mathfrak{d}f \otimes v) &= (d_*[u, f] - [d_*u, f] - [u, d_*f]) \wedge v, \\ \mathcal{D}(\hat{d}_*)(\mathfrak{d}f \otimes u, \mathfrak{d}g \otimes v) &= ([d_*f, g] + [f, d_*g]) \wedge u \wedge v. \end{aligned}$$

which implies that  $(E, E^*)$  is a Lie bialgebroid if and only if  $\hat{d}_*$  is closed.

It is clear that  $d_* = [\tau, \cdot] \iff \hat{d}_* = \mathcal{D}\tau$ , which implies (2). ■

Finally we consider the deformation of a projective Lie algebroid  $A$  and its lifted Dirac structure  $L^A$ . Let  $\Omega : A \wedge A \rightarrow A \cap E$  be a bundle map. Consider an  $\varepsilon$ -parameterized family of brackets

$$[a, b]_A^\varepsilon = [a, b]_A + \varepsilon\Omega(a, b), \quad \forall a, b \in \Gamma(A).$$

If every  $\varepsilon$ -bracket endows  $A$  a projective Lie algebroid structure, we say that  $\Omega$  generates a deformation of the projective Lie algebroid  $A$ . Evidently, this requirement is equivalent to the following compatibility conditions:

$$\Omega([a, b]_A, c) + [\Omega(a, b), c]_A + c.p. = 0, \quad (26)$$

$$\Omega(\Omega(a, b), c) + c.p. = 0. \quad (27)$$

Equation (27) means that  $\Omega$  itself defines a (fibrewise) Lie bracket. Furthermore, for all  $l_1, l_2, l_3 \in \Gamma(L^A)$ , we have

$$\begin{aligned} d_{L^A} \mathbf{b}^* \Omega(l_1, l_2, l_3) &= \rho_{L^A}(l_1) \mathbf{b}^* \Omega(l_2, l_3) + c.p. + \mathbf{b}^* \Omega(\{l_1, l_2\}, l_3) + c.p. \\ &= \rho_{L^A}(l_1) \Omega(\mathbf{b}l_2, \mathbf{b}l_3) + c.p. + \Omega(\mathbf{b}\{l_1, l_2\}, \mathbf{b}l_3) + c.p. \\ &= \mathbf{b}\{l_1, \mathbf{d}\Omega(\mathbf{b}l_2, \mathbf{b}l_3)\} + c.p. + \Omega([\mathbf{b}l_1, \mathbf{b}l_2]_A, \mathbf{b}l_3) + c.p. \\ &= [\mathbf{b}l_1, \Omega(\mathbf{b}l_2, \mathbf{b}l_3)]_A + c.p. + \Omega([\mathbf{b}l_1, \mathbf{b}l_2]_A, \mathbf{b}l_3) + c.p., \end{aligned}$$

which implies that Equation (26) is equivalent to the requirement that  $\mathbf{b}^* \Omega$  is closed.

Since there is a one-to-one correspondence between reducible Dirac structures and projective Lie algebroids, we can associate a deformation of the Dirac structure  $L^A$  to the deformation of the projective Lie algebroid  $A$ . Denote the deformed projective Lie algebroid by  $A_\varepsilon$ , then the deformed Dirac structure  $L^{A_\varepsilon}$  is give by

$$L^{A_\varepsilon} = \{l + h \mid l \in L_A, h \in \text{Hom}(\mathcal{T}, E), \text{ s.t., } h(a) = \Omega(\mathbf{b}(l), a), \quad \forall a \in A\}.$$

An interesting problem is to consider a deformation  $\Omega$  which is a coboundary:

$$\mathbf{b}^* \Omega = d_{L^A} \omega_X, \quad \text{for some } X \in \Gamma(\mathcal{E}). \quad (28)$$

**Proposition 6.5.** *Let  $\Omega : A \wedge A \longrightarrow A \cap E$  be a bundle map. If  $\mathbf{b}^* \Omega = d_{L^A} \omega_X$  for some  $X \in \Gamma(\mathcal{E})$ , then  $X \in N_{A^0} = N_{\mathbf{b}^{-1}(A)}$ . Moreover, we have*

$$\Omega(a_1, a_2) = \frac{1}{2}([X_\bullet a_1, a_2]_A + [a_1, X_\bullet a_2]_A - X_\bullet[a_1, a_2]_A), \quad \forall a_1, a_2 \in \Gamma(A). \quad (29)$$

Furthermore,  $\Omega$  generates a deformation of the projective Lie algebroid  $A$  if and only if

$$[T^X(a, b), c]_A + T^X([a, b]_A, c) + c.p. = 0, \quad \forall a, b, c \in \Gamma(A), \quad (30)$$

where  $T^X : \Gamma(A) \wedge \Gamma(A) \rightarrow \Gamma(A)$  is defined by

$$T^X(a, b) \triangleq X_\bullet([X_\bullet a, b]_A + [a, X_\bullet b]_A - X_\bullet[a, b]_A) - [X_\bullet a, X_\bullet b]_A.$$

Conversely, for any  $X \in N_{\mathbf{b}^{-1}(A)} = N_{A^0}$  satisfying (30),  $\Omega$  defined by equation (29) is a bundle map from  $A \wedge A$  to  $A \cap E$  that generates a deformation of  $A$  and relation (28) holds.

**Proof.** By Equations (23) and (28), for all  $\theta \in \Gamma(A^0)$ ,  $l \in \Gamma(L^A)$ , we have

$$0 = (\mathbf{b}^* \Omega)(\theta, l) = d_{L^A} \omega_X(\theta, l) = -(\{X, \theta\}, l)_E.$$

Thus,  $\{X, \theta\} \in \Gamma(L^A \cap \text{Hom}(\mathcal{T}, E)) = \Gamma(A^0)$ , i.e.  $X \in N_{A^0}$ . For any  $Y \in \Gamma(\mathbf{b}^{-1}(A))$ , we have

$$\begin{aligned} \{X, \theta\}(\mathbf{b}Y) &= 2(\{X, \theta\}, Y)_E = 2\rho(X)(\theta, Y)_E - 2(\theta, \{X, Y\})_E \\ &= \rho(X)\theta(\mathbf{b}Y) - \theta \circ \mathbf{b}\{X, Y\}, \end{aligned}$$

which implies that  $\theta \circ \mathbf{b}\{X, Y\} = 0$ , i.e.  $X \in N_{\mathbf{b}^{-1}(A)}$ .

Let  $l_i \in \Gamma(L^A)$  and  $\mathbf{b}(l_i) = a_i$ . By some straightforward computation, we have

$$\begin{aligned}
\Omega(a_1, a_2) &= (\mathbf{b}^*\Omega)(l_1, l_2) = d_{L^A}\omega_X(l_1, l_2) \\
&= -(\{X, l_1\}, l_2)_E = -\frac{1}{2}\mathbf{b}(\{\{X, l_1\}, l_2\} + \{l_2, \{X, l_1\}\}) \\
&= -\frac{1}{2}\mathbf{b}(\{X, \{l_1, l_2\}\} - \{l_1, \{X, l_2\}\} + \{l_2, \{X, l_1\}\}) \\
&= \frac{1}{2}([X_\bullet a_1, a_2]_A + [a_1, X_\bullet a_2]_A - X_\bullet[a_1, a_2]_A),
\end{aligned}$$

which implies Equation (29).

If  $\Omega$  generates a deformation of the projective Lie algebroid  $A$ ,  $\Omega$  itself defines a fibrewise Lie bracket. It is easy to see that this is equivalent to (30). The other conclusions can be easily checked. ■

For a Lie algebroid  $(\mathcal{A}, [\cdot, \cdot], \alpha)$ , a Nijenhuis operator is a bundle map  $N : \mathcal{A} \rightarrow \mathcal{A}$  such that the following equality holds

$$T^N(a, b) \triangleq N([Na, b] + [a, Nb] - N[a, b]) - [Na, Nb] = 0, \quad \forall a, b \in \Gamma(\mathcal{A}).$$

It induces a new Lie algebroid  $(\mathcal{A}, [\cdot, \cdot]_N, \alpha_N)$ , where  $\alpha_N = \alpha \circ N$  and

$$[a, b]_N = [Na, b] + [a, Nb] - N[a, b].$$

In fact,  $T^N = 0$  is only a sufficient condition for the bracket operation  $[\cdot, \cdot]_N$  being a Lie bracket. The necessary and sufficient condition is

$$[a, T^N(b, c)] + T^N(a, [b, c]) + c.p. = 0.$$

The role of the operator  $X_\bullet : \Gamma(A) \rightarrow \Gamma(A)$  is just like that of a Nijenhuis operator. In general,  $X_\bullet$  is not a bundle map, but it still induces a twist of the Lie algebroid. In fact,  $X_\bullet$  is a bundle map if and only if  $X \in \Gamma(\text{Hom}(\mathcal{T}, E))$  and in this case  $X_\bullet = X|_A : A \rightarrow A \cap E$ , which is a Nijenhuis operator if  $T^X$  vanishes.

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